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Non-canonical quantum optics

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Abstract. Modification of the right-hand-side of canonical commutation relations (CCR) naturally occurs if one considers a harmonic oscillator with indefinite frequency. Quantization of the electromagnetic field by means of such a non-CCR algebra automatically removes the infinite energy the of vacuum but still results in a theory which is very similar to quantum electrodynamics. An analysis of perturbation theory shows that the non-canonical theory has an automatically built-in cut-off but requires charge/mass renormalization already at the non-relativistic level. A simple rule allowing one to compare perturbative predictions of canonical and non-canonical theories is given. The notion of a unique vacuum state is replaced by a set of different vacua. Multi-photon states are defined in the standard way but depend on the choice of vacuum. Making a simplified choice of the vacuum state we estimate corrections to atomic lifetimes, probabilities of multiphoton spontaneous and stimulated emission and Planck's law. The results are very similar to the standard ones but there are also differences which can be tested experimentally. Two different candidates for a free-field Hamiltonian are compared.

1. Introduction

The standard quantization of a harmonic oscillator is based on quantization of p and q but with ω as a parameter. To have, say, two different frequencies one has to consider two independent oscillators. On the other hand, it is evident that there can exist oscillators which are in a *quantum superposition* of different frequencies. The example is an oscillator wavepacket associated with the distribution of centre-of-mass momenta. It is known that the superposition of momenta becomes translated into a superposition of Doppler shifts and therefore also of frequencies (cf [1]). We stress here the word 'quantum' since the superpositions we have in mind are not those we know from *classical* oscillations.

This trivial observation raises the question of the role of superpositions of frequencies for the description of a single harmonic oscillator. The motivation behind the problem is associated with the question of field quantization: is it possible that a quantum field consists of oscillators whose frequencies are *indefinite*? If so, maybe to quantize the field it is sufficient to use only one oscillator which exists in a *quantum* superposition of all the possible frequencies allowed by the boundary conditions of a given problem?

The idea is very simple. It is known that a 'one-particle' state vector can be regarded as a representation of an ensemble of particles in a given pure state. On the other hand, the classical electromagnetic field can be regarded as an ensemble of oscillators. The standard idea of quantization, going back to 1925 [2], is to treat the field as an ensemble of quantum

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oscillators. However, the ensemble itself is, in a sense, a classical one since for each frequency we need a separate oscillator. This is analogous to a classical ensemble of particles forming a classical wave on a lake. For each point at its surface we need a separate particle because a classical particle can occupy only a single point in space. A quantum wave is of course different and we are all accustomed to the idea of a single-particle wave. In this case the properties of the entire ensemble are somehow encoded in properties of a single element of the ensemble.

For some reasons, probably partly historical, it seems that the idea of a single-particle state vector representation of the ensemble of oscillators has never been considered. The historical reason may be the fact that the very concept of field quantization occurred already in 1925. At that stage quantum mechanics still existed in a matrix form and the Schrödinger paper 'Quantizierung als Eigenwertproblem' [3], where the Schrödinger equation occurred for the first time and the role of eigenvalues was explained, was not yet published. Actually, as explained by Jammer [4], Heisenberg quantized the harmonic oscillator not having heard of matrices so no wonder he could not treat the parameter ω entering $E = p^2/2m + m\omega^2 q^2/2$ as an eigenvalue of some operator.

Heisenberg's quantization leads to the well known algebra of canonical commutation relations (CCR) $[a_\omega, a_\omega^\dagger] = \mathbf{1}$, where $\mathbf{1}$ is the identity operator and ω is a classical parameter. As it turns out the replacement of ω by an operator $\hat{\omega}$ leads to non-canonical commutation relations (non-CCR) $[\boldsymbol{a}_\omega, \boldsymbol{a}_\omega^\dagger] = \hat{\mathbf{1}}_\omega$, where ω is an eigenvalue of $\hat{\omega}$ and $\sum_\omega \hat{\mathbf{1}}_\omega = \mathbf{1}$. $\hat{\mathbf{1}}_\omega$, similarly to $\mathbf{1}$ commutes with all creation and annihilation operators and the remaining commutators of non-CCR are the same as those of CCR. This subtle difference of the right-hand sides of CCR and non-CCR immediately explains why a non-CCR-quantized electromagnetic field will have a vacuum with finite energy. Since the non-CCR and CCR algebras are so similar to each other it is not surprising that the resulting theories are also very similar.

The main consequence of the non-CCR modification of CCR is a different normalization of n-photon states. For example,

$$\sum_\omega \langle \mathbf{0}|\boldsymbol{a}_\omega \boldsymbol{a}_\omega^\dagger|\mathbf{0}\rangle = 1 \tag{1}$$

and therefore $\langle \mathbf{0}|\boldsymbol{a}_\omega \boldsymbol{a}_\omega^\dagger|\mathbf{0}\rangle < 1$ (the boldface font is used for the non-CCR-quantized objects). This should be contrasted with the CCR result $\langle 0|a_\omega a_\omega^\dagger|0\rangle = 1$, and the resulting divergence

$$\sum_\omega \langle 0|a_\omega a_\omega^\dagger|0\rangle = \infty \tag{2}$$

which is *the* source of the infinite vacuum energy.

To end these introductory remarks one should mention that several approaches towards an alternative description of the electromagnetic field at a fundamental level have already been proposed (e.g. Janes' [5] neoclassical theory or stochastic electrodynamics [6]). However, the main idea of all such alternatives was to treat the field in classical terms and to associate the observed discreteness of emission/absorption phenomena with the quantum nature of atoms and not with the field itself.

The approach we will discuss in this paper does not belong to this tradition, is much more radical and, so to say, goes in the opposite direction. We will not try to make the field more classical. What we will try to do is to make it even *more quantum* by replacing classical parameters with eigenvalues.

2. Harmonic oscillator in a superposition of frequencies

We know that frequency is typically associated with an eigenvalue of some Hamiltonian or, which is basically the same, with boundary conditions. A natural way of incorporating different frequencies into a single harmonic oscillator is by means of a *frequency operator*

$$\Omega = \sum_{\omega_k, j_k} \omega_k |\omega_k, j_k\rangle\langle\omega_k, j_k| \tag{3}$$

where all $\omega_k \geqslant 0$. For simplicity we have limited the discussion to the discrete spectrum but it is useful to include from the outset the possibility of degeneracies, represented here by the additional discrete quantum numbers j_k. Ω is an unbounded operator acting in a Hilbert space $\mathcal{H}_1$ and its domain D_Ω consists of vectors ψ satisfying $\langle\psi|\Omega|\psi\rangle < \infty$. The corresponding Hamiltonian is defined by

$$H = \hbar\Omega \otimes \tfrac{1}{2}\big(a^\dagger a + aa^\dagger\big) \tag{4}$$

where $a = \sum_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1|$. H acts in $D_\Omega \otimes \mathcal{H}_{\mathrm{HO}}$, where $\mathcal{H}_{\mathrm{HO}}$ is the Hilbert space of states of a harmonic oscillator whose frequency is $\omega = 1$. The eigenstates of H are $|\omega_k, j_k, n\rangle$ and satisfy the required formula

$$H|\omega_k, j_k, n\rangle = \hbar\omega_k\big(n + \tfrac{1}{2}\big)|\omega_k, j_k, n\rangle \tag{5}$$

justifying our choice of H. The standard case of the oscillator whose frequency is just ω corresponds either to $\Omega = \omega\mathbf{1}$, where $\mathbf{1}$ is the identity operator in $\mathcal{H}_1$, or to the subspace spanned by $|\omega_k, j_k, n\rangle$ with fixed $\omega_k = \omega$. Introducing the operators

$$a_{\omega_k, j_k} = |\omega_k, j_k\rangle\langle\omega_k, j_k| \otimes a \tag{6}$$

we find that

$$H = \tfrac{1}{2}\sum_{\omega_k, j_k} \hbar\omega_k\big(a^\dagger_{\omega_k, j_k} a_{\omega_k, j_k} + a_{\omega_k, j_k} a^\dagger_{\omega_k, j_k}\big). \tag{7}$$

The algebra of the oscillator is 'non-canonical':

$$[a_{\omega_k, j_k}, a^\dagger_{\omega_k, j_k}] = |\omega_k, j_k\rangle\langle\omega_k, j_k| \otimes \mathbf{1} = 1_{\omega_k, j_k} \tag{8}$$

$$[a_{\omega_k, j_k}, a^\dagger_{\omega_l, j_l}] = 0 \qquad \text{for} \quad (\omega_k, j_k) \neq (\omega_l, j_l) \tag{9}$$

$$[a_{\omega_k, j_k}, a_{\omega_l, j_l}] = 0 \tag{10}$$

$$[a^\dagger_{\omega_k, j_k}, a^\dagger_{\omega_l, j_l}] = 0. \tag{11}$$

The dynamics in the Schrödinger picture is given by

$$\mathrm{i}\hbar\partial_t|\Psi\rangle = H|\Psi\rangle = \hbar\Omega \otimes \big(a^\dagger a + \tfrac{1}{2}\mathbf{1}\big)|\Psi\rangle. \tag{12}$$

In the Heisenberg picture we obtain the important formula

$$a_{\omega_k, j_k}(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar} a_{\omega_k, j_k} \mathrm{e}^{-\mathrm{i}Ht/\hbar} \tag{13}$$

$$= |\omega_k, j_k\rangle\langle\omega_k, j_k| \otimes \mathrm{e}^{-\mathrm{i}\omega_k t} a = \mathrm{e}^{-\mathrm{i}\omega_k t} a_{\omega_k, j_k}. \tag{14}$$

Taking a general state

$$|\psi\rangle = \sum_{\omega_k, j_k, n} \psi(\omega_k, j_k, n)|\omega_k, j_k, n\rangle \tag{15}$$

we find that the average energy of the oscillator is

$$\langle H\rangle = \langle\psi|H|\psi\rangle = \sum_{\omega_k, j_k, n} |\psi(\omega_k, j_k, n)|^2 \hbar\omega_k\left(n + \tfrac{1}{2}\right). \tag{16}$$

The average clearly looks as an average energy of an *ensemble of different and independent oscillators*. The 'vacuum state' of the ensemble, i.e. the one with $\psi(\omega_k, j_k, n > 0) = 0$ has the energy

$$\langle H\rangle = \tfrac{1}{2}\sum_{\omega_k, j_k} |\psi(\omega_k, j_k, 0)|^2 \hbar\omega_k \tag{17}$$

which is finite if

$$\sum_{\omega_k, j_k} \psi(\omega_k, j_k, 0)|\omega_k, j_k\rangle \tag{18}$$

belongs to the domain of Ω. In analogy with the ensemble of independent harmonic oscillators one can consider states resembling several harmonic oscillators with different frequencies and different excitations for each of the frequencies. For example,

$$|\psi\rangle = \psi_1|\omega_1, j, 3\rangle + \psi_2|\omega_2, j, 5\rangle \tag{19}$$

represents a 'coherent mixture' of two oscillators with the same internal state j, one having frequency ω_1 and being in the third excited state, the other having frequency ω_2 and excited to the fifth state. The one-to-one correlation between the frequency and the number of excitations is possible due to the fact that (19) is an *entangled state* in $\mathcal{H}_1 \otimes \mathcal{H}_{\mathrm{HO}}$.

Let us also note that what we have called the 'vacuum state' is not the ground energy state of the Hamiltonian, the latter being

$$\sum_j |\omega_{\min}, j, 0\rangle \tag{20}$$

where $\omega_{\min}$ is the smallest eigenvalue of Ω. The 'vacuum state' is the state where all the oscillators of the coherent ensemble are in their ground states. The ground state corresponds to the situation where the ensemble consists of oscillators of only one type, namely those with the minimal frequency $\omega_{\min}$. The vacuum state is neither the lowest-energy state of the Hamiltonian nor even its eigenstate. However, on this basis one should not conclude that the vacuum is unstable. To make it unstable the Hamiltonian should have non-vanishing matrix elements between states of different frequencies, which is not the case.

Although the above properties of the indefinite-frequency harmonic oscillator are not in themselves very surprising, they are still quite remarkable if one thinks of the problem of field quantization.

The very idea of quantizing the electromagnetic field, as put forward by Born *et al* [2] and Dirac [7], is based on the observation that the mode decomposition of the electromagnetic energy is analogous to the energy of an ensemble of independent harmonic oscillators. In 1925, after the work of Heisenberg, it was clear what to do: one had to replace each classical oscillator by a quantum one. However, since each oscillator had a definite frequency, to have an infinite number of different frequencies one needed an infinite number of oscillators. The price one paid for this assumption was the infinite energy of the electromagnetic vacuum.

The infinity is regarded as an 'easy' one since one can get rid of it by redefining the Hamiltonian and removing the infinite term. The result looks correct and many properties typical of a *quantum* harmonic oscillator are indeed observed in the electromagnetic field. However, subtraction of infinite terms is in mathematics as forbidden as division by zero

so to avoid evident absurdities one is forced to invent various *ad hoc* regularizations whose only justification is that otherwise the theory would not work. In a larger perspective (say, in cosmology) it is not at all clear that an infinite (or arbitrarily cut off at the Planck scale) energy of the vacuum does not lead to contradictions with observational data [8]. Finally, Dirac himself had never been fully satisfied by the theory he created. As Weinberg put it, Dirac's 'demand for a completely finite theory is similar to a host of other aesthetic judgements that theoretical physicists always need to make' [12].

The oscillator that can exist in superpositions of different frequencies is a natural candidate as a starting point for Dirac-type field quantization. Symbolically, if the Heisenberg quantization is $p^2 + \omega^2 q^2 \mapsto \hat{p}^2 + \omega^2 \hat{q}^2$, where ω is a parameter, the new scheme is $p^2 + \omega^2 q^2 \mapsto \hat{p}^2 + \hat{\omega}^2 \hat{q}^2$, where $\hat{\omega}$ is an operator. Its spectrum can be related to boundary conditions imposed on the fields.

We do not need to remove the vacuum energy since in the Hilbert space of physical states the correction is finite. The question we have to understand is whether one can obtain the well known quantum properties of the radiation field by this type of quantization.

3. 'First quantization'—one-oscillator field operators

The new quantization will be performed in two steps. In this section we describe the first step, a kind of first quantization. In next sections we shall perform an analogue of second quantization which will lead to the final framework.

The energy and momentum operators of the field are defined in analogy with H from the previous section,

$$H = \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} |s,\vec{\kappa}\rangle\langle s,\vec{\kappa}| \otimes \tfrac{1}{2}\left(a^\dagger a + aa^\dagger\right) \tag{21}$$

$$= \tfrac{1}{2} \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} \left(a^\dagger_{s,\vec{\kappa}} a_{s,\vec{\kappa}} + a_{s,\vec{\kappa}} a^\dagger_{s,\vec{\kappa}}\right) \tag{22}$$

$$\vec{P} = \sum_{s,\vec{\kappa}} \hbar\vec{\kappa} |s,\vec{\kappa}\rangle\langle s,\vec{\kappa}| \otimes \tfrac{1}{2}\left(a^\dagger a + aa^\dagger\right) \tag{23}$$

$$= \tfrac{1}{2} \sum_{s,\vec{\kappa}} \hbar\vec{\kappa} \left(a^\dagger_{s,\vec{\kappa}} a_{s,\vec{\kappa}} + a_{s,\vec{\kappa}} a^\dagger_{s,\vec{\kappa}}\right) \tag{24}$$

where $s = \pm 1$ corresponds to circular polarizations. Denote $P = (H/c, \vec{P})$ and $P \cdot x = Ht - \vec{P} \cdot \vec{x}$. We employ the standard Dirac-type definitions for mode quantization in volume V

$$\hat{\vec{A}}(t,\vec{x}) = \sum_{s,\vec{\kappa}} \sqrt{\frac{\hbar}{2\omega_{\vec{\kappa}} V}} \left(a_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \vec{e}_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}} + a^\dagger_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\omega_{\vec{\kappa}} t} \vec{e}^{\,*}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}}\right) \tag{25}$$

$$= \mathrm{e}^{\mathrm{i}P\cdot x/\hbar} \hat{\vec{A}} \mathrm{e}^{-\mathrm{i}P\cdot x/\hbar} \tag{26}$$

$$\hat{\vec{E}}(t,\vec{x}) = \mathrm{i} \sum_{s,\vec{\kappa}} \sqrt{\frac{\hbar\omega_{\vec{\kappa}}}{2V}} \left(a_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}_{s,\vec{\kappa}} - a^\dagger_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}^{\,*}_{s,\vec{\kappa}}\right) \tag{27}$$

$$= \mathrm{e}^{\mathrm{i}P\cdot x/\hbar} \hat{\vec{E}} \mathrm{e}^{-\mathrm{i}P\cdot x/\hbar} \tag{28}$$

$$\hat{\vec{B}}(t,\vec{x}) = \mathrm{i} \sum_{s,\vec{\kappa}} \sqrt{\frac{\hbar\omega_{\vec{\kappa}}}{2V}} \vec{n}_\kappa \times \left(a_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}_{s,\vec{\kappa}} - a^\dagger_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}^{\,*}_{s,\vec{\kappa}}\right) \tag{29}$$

$$= \mathrm{e}^{\mathrm{i}P\cdot x/\hbar}\hat{\vec{B}}\mathrm{e}^{-\mathrm{i}P\cdot x/\hbar} \tag{30}$$

where

$$a_{s,\vec{\kappa}} = |s,\vec{\kappa}\rangle\langle s,\vec{\kappa}| \otimes a \tag{31}$$

$$a^{\dagger}_{s,\vec{\kappa}} = |s,\vec{\kappa}\rangle\langle s,\vec{\kappa}| \otimes a^{\dagger}. \tag{32}$$

For later purposes we introduce the notation

$$[a_{s,\vec{\kappa}}, a^{\dagger}_{s,\vec{\kappa}}] = 1_{s,\vec{\kappa}} = |s,\vec{\kappa}\rangle\langle s,\vec{\kappa}| \otimes \mathbf{1}. \tag{33}$$

Now take a state (say, in the Heisenberg picture)

$$|\Psi\rangle = \sum_{s,\vec{\kappa},n} \Psi_{s,\vec{\kappa},n}|s,\vec{\kappa},n\rangle \tag{34}$$

$$= \sum_{s,\vec{\kappa}} \Phi_{s,\vec{\kappa}}|s,\vec{\kappa}\rangle|\alpha_{s,\vec{\kappa}}\rangle \tag{35}$$

where $|\alpha_{s,\vec{\kappa}}\rangle$ form a family of one-oscillator coherent states:

$$a|\alpha_{s,\vec{\kappa}}\rangle = \alpha_{s,\vec{\kappa}}|\alpha_{s,\vec{\kappa}}\rangle. \tag{36}$$

The averages of the field operators are

$$\langle\Psi|\hat{\vec{A}}(t,\vec{x})|\Psi\rangle = \sum_{s,\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2 \sqrt{\frac{\hbar}{2\omega_{\vec{\kappa}}V}}\left(\alpha_{s,\vec{\kappa}}\mathrm{e}^{-\mathrm{i}\kappa\cdot x}\vec{e}_{s,\vec{\kappa}} + \alpha^{*}_{s,\vec{\kappa}}\mathrm{e}^{\mathrm{i}\kappa\cdot x}\vec{e}^{\,*}_{s,\vec{\kappa}}\right) \tag{37}$$

$$\langle\Psi|\hat{\vec{E}}(t,\vec{x})|\Psi\rangle = \sum_{s,\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2 \sqrt{\frac{\hbar\omega_{\vec{\kappa}}}{2V}}\left(\alpha_{s,\vec{\kappa}}\mathrm{e}^{-\mathrm{i}\kappa\cdot x}\vec{e}_{s,\vec{\kappa}} - \alpha^{*}_{s,\vec{\kappa}}\mathrm{e}^{\mathrm{i}\kappa\cdot x}\vec{e}^{\,*}_{s,\vec{\kappa}}\right) \tag{38}$$

$$\langle\Psi|\hat{\vec{B}}(t,\vec{x})|\Psi\rangle = \mathrm{i}\sum_{s,\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2 \sqrt{\frac{\hbar\omega_{\vec{\kappa}}}{2V}}\left(\alpha_{s,\vec{\kappa}}\mathrm{e}^{-\mathrm{i}\kappa\cdot x}\vec{n}_{\kappa}\times\vec{e}_{s,\vec{\kappa}} - \alpha^{*}_{s,\vec{\kappa}}\mathrm{e}^{\mathrm{i}\kappa\cdot x}\vec{n}_{\kappa}\times\vec{e}^{\,*}_{s,\vec{\kappa}}\right). \tag{39}$$

These are just the classical fields. More precisely, the fields look like monochromatic coherent states averaged with respect to probabilities $|\Phi_{s,\vec{\kappa}}|^2$. The energy–momentum operators also satisfy the standard relations

$$H = \tfrac{1}{2}\int_V \mathrm{d}^3x\left(\hat{\vec{E}}(t,\vec{x})\cdot\hat{\vec{E}}(t,\vec{x}) + \hat{\vec{B}}(t,\vec{x})\cdot\hat{\vec{B}}(t,\vec{x})\right) \tag{40}$$

$$\vec{P} = \int_V \mathrm{d}^3x\,\hat{\vec{E}}(t,\vec{x})\times\hat{\vec{B}}(t,\vec{x}). \tag{41}$$

To end this section let us note that

$$\langle\Psi|H|\Psi\rangle = \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}}|\Phi_{s,\vec{\kappa}}|^2\left(|\alpha_{s,\vec{\kappa}}|^2 + \tfrac{1}{2}\right) \tag{42}$$

$$\langle\Psi|\vec{P}|\Psi\rangle = \sum_{s,\vec{\kappa}} \hbar\vec{\kappa}|\Phi_{s,\vec{\kappa}}|^2\left(|\alpha_{s,\vec{\kappa}}|^2 + \tfrac{1}{2}\right). \tag{43}$$

The contribution from the vacuum fluctuations is non-zero but *finite*. One can also phrase the latter property as follows. The non-canonical algebra of creation–annihilation operators satisfies the resolution of identity

$$\sum_{s,\vec{\kappa}}[a_{s,\vec{\kappa}}, a^{\dagger}_{s,\vec{\kappa}}] = \mathbf{1} \tag{44}$$

whereas the canonical algebra would imply

$$\sum_{s,\vec{\kappa}}[a_{s,\vec{\kappa}}, a^{\dagger}_{s,\vec{\kappa}}] = \infty\mathbf{1}. \tag{45}$$

4. 'Second quantization'

The Hilbert space of states of the field we have constructed is spanned by vectors $|s, \vec{\kappa}, n\rangle$. Still there is no doubt that both in reality and the standard formalism there exist multiparticle entangled states such as those spanned by tensor products of the form

$$|+, \vec{\kappa}_1, 1\rangle|-, \vec{\kappa}_2, 1\rangle \tag{46}$$

and similar. It seems that there is no reason to limit our discussion to a *single* Hilbert space of a *single* oscillator. What we have done so far was a quantization of the electromagnetic field at the level of a 'one-particle' Hilbert space. Similarly to quantization of other physical systems the next step is to consider many particles. What is not obvious (physically) is whether the oscillators should be considered as non-interacting. This physical freedom leads to two natural candidates for a free-field Hamiltonian.

The *non-interacting* extension is essentially clear. Having the one-particle energy–momentum operators P_a (i.e. generators of 4-translations in the one-particle Hilbert space) we define in the standard way their extensions to the Fock-type space

$$\mathcal{P}_a = P_a \oplus \big(P_a \otimes \mathbf{1} + \mathbf{1} \otimes P_a\big) \oplus \big(P_a \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes P_a \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes P_a\big) \oplus \cdots. \tag{47}$$

The x dependence of fields is introduced similarly to the one-particle level

$$\vec{\mathcal{F}}(t, \vec{x}) = \mathrm{e}^{\mathrm{i}\mathcal{P}\cdot x/\hbar}\vec{\mathcal{F}}\mathrm{e}^{-\mathrm{i}\mathcal{P}\cdot x/\hbar} \tag{48}$$

but the field itself has yet to be defined. Assume

$$\vec{\mathcal{F}} = c_1\vec{F} \oplus c_2\big(\vec{F} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{F}\big) \oplus c_3\big(\vec{F} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{F} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \vec{F}\big) \oplus \cdots \tag{49}$$

where c_k are constants discussed below, and $\vec{F}$ is $\hat{\vec{A}}$, $\hat{\vec{E}}$ or $\hat{\vec{B}}$. The multi-oscillator annihilation operator associated with such fields must be therefore of the form

$$\begin{aligned}\boldsymbol{a}_{s,\vec{\kappa}} &= c_1 a_{s,\vec{\kappa}} \oplus c_2\big(a_{s,\vec{\kappa}} \otimes \mathbf{1} + \mathbf{1} \otimes a_{s,\vec{\kappa}}\big) \\ &\quad \oplus c_3\big(a_{s,\vec{\kappa}} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes a_{s,\vec{\kappa}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes a_{s,\vec{\kappa}}\big) \oplus \cdots.\end{aligned} \tag{50}$$

Having two one-particle operators, say X and Y, one can easily establish a relation between the one-particle commutator $[X, Y]$ and the commutator of the extensions $\mathcal{X}$, $\mathcal{Y}$:

$$\begin{aligned}[\mathcal{X}, \mathcal{Y}] &= c_1^2[X, Y] \oplus c_2^2\big([X, Y] \otimes \mathbf{1} + \mathbf{1} \otimes [X, Y]\big) \\ &\quad \oplus c_3^2\big([X, Y] \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes [X, Y] \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes [X, Y]\big) \oplus \cdots.\end{aligned} \tag{51}$$

The annihilation operators thus defined therefore satisfy the algebra

$$[\boldsymbol{a}_{s,\vec{\kappa}}, \boldsymbol{a}^\dagger_{s',\vec{\kappa}\,'}] = 0 \qquad \text{for} \quad (s, \vec{\kappa}) \neq (s', \vec{\kappa}\,') \tag{52}$$

$$[\boldsymbol{a}_{s,\vec{\kappa}}, \boldsymbol{a}^\dagger_{s,\vec{\kappa}}] = \hat{\mathbf{1}}_{s,\vec{\kappa}} \tag{53}$$

$$[\boldsymbol{a}_{s,\vec{\kappa}}, \boldsymbol{a}_{s',\vec{\kappa}\,'}] = 0 \tag{54}$$

$$[\boldsymbol{a}^\dagger_{s,\vec{\kappa}}, \boldsymbol{a}^\dagger_{s',\vec{\kappa}\,'}] = 0 \tag{55}$$

where the operator $\hat{\mathbf{1}}_{s,\vec{\kappa}}$ is defined by

$$\begin{aligned}\hat{\mathbf{1}}_{s,\vec{\kappa}} &= c_1^2 1_{s,\vec{\kappa}} \oplus c_2^2\big(1_{s,\vec{\kappa}} \otimes \mathbf{1} + \mathbf{1} \otimes 1_{s,\vec{\kappa}}\big) \\ &\quad \oplus c_3^2\big(1_{s,\vec{\kappa}} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes 1_{s,\vec{\kappa}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes 1_{s,\vec{\kappa}}\big) \oplus \cdots\end{aligned} \tag{56}$$

and $1_{s,\vec{\kappa}}$ is a single-oscillator operator (33).

An important property of the one-oscillator description was the resolution of identity (44). The requirement that the same be valid at the multi oscillator level leads to $c_n = 1/\sqrt{n}$. In such a case one finds that

$$\hat{\mathbf{1}}^2_{s,\vec{\kappa}} \neq \hat{\mathbf{1}}_{s,\vec{\kappa}} \tag{57}$$

but nevertheless

$$\sum_{s,\vec{\kappa}} \hat{\mathbf{1}}_{s,\vec{\kappa}} = \mathbf{1}. \tag{58}$$

That is $\hat{\mathbf{1}}_{s,\vec{\kappa}}$ are the so-called positive-operator-valued (POV) measures [9]. Below we shall give another justification of this particular choice of c_n.

We can finally write

$$\vec{\mathcal{A}}(t,\vec{x}) = \sum_{s,\vec{\kappa}} \sqrt{\frac{\hbar}{2\omega_{\vec{\kappa}} V}} \left(\boldsymbol{a}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \vec{e}_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}} + \boldsymbol{a}^\dagger_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\omega_{\vec{\kappa}} t} \vec{e}^{\,*}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}}\right) \tag{59}$$

$$= \mathrm{e}^{\mathrm{i}\mathcal{P}\cdot x/\hbar} \vec{\mathcal{A}} \mathrm{e}^{-\mathrm{i}\mathcal{P}\cdot x/\hbar} \tag{60}$$

$$\vec{\mathcal{E}}(t,\vec{x}) = \mathrm{i} \sum_{s,\vec{\kappa}} \sqrt{\frac{\hbar\omega_{\vec{\kappa}}}{2V}} \left(\boldsymbol{a}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}_{s,\vec{\kappa}} - \boldsymbol{a}^\dagger_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}^{\,*}_{s,\vec{\kappa}}\right) \tag{61}$$

$$= \mathrm{e}^{\mathrm{i}\mathcal{P}\cdot x/\hbar} \vec{\mathcal{E}} \mathrm{e}^{-\mathrm{i}\mathcal{P}\cdot x/\hbar} \tag{62}$$

$$\vec{\mathcal{B}}(t,\vec{x}) = \mathrm{i} \sum_{s,\vec{\kappa}} \sqrt{\frac{\hbar\omega_{\vec{\kappa}}}{2V}} \vec{n}_\kappa \times \left(\boldsymbol{a}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}_{s,\vec{\kappa}} - \boldsymbol{a}^\dagger_{s,\vec{\kappa}} \mathrm{e}^{\mathrm{i}\omega_{\vec{\kappa}} t} \mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}} \vec{e}^{\,*}_{s,\vec{\kappa}}\right) \tag{63}$$

$$= \mathrm{e}^{\mathrm{i}\mathcal{P}\cdot x/\hbar} \vec{\mathcal{B}} \mathrm{e}^{-\mathrm{i}\mathcal{P}\cdot x/\hbar}. \tag{64}$$

These operators form a basis of the modified version of non-relativistic quantum optics.

Let us return for the moment to the case of a general c_n. A straightforward calculation shows that

$$\boldsymbol{H} = \tfrac{1}{2}\int_V \mathrm{d}^3x \left(\vec{\mathcal{E}}(t,\vec{x})\cdot\vec{\mathcal{E}}(t,\vec{x}) + \vec{\mathcal{B}}(t,\vec{x})\cdot\vec{\mathcal{B}}(t,\vec{x})\right) = \tfrac{1}{2}\sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} \left(\boldsymbol{a}^\dagger_{s,\vec{\kappa}} \boldsymbol{a}_{s,\vec{\kappa}} + \boldsymbol{a}_{s,\vec{\kappa}} \boldsymbol{a}^\dagger_{s,\vec{\kappa}}\right) \tag{65}$$

$$\begin{aligned}
&= \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} \big[c_1^2 \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \\
&\oplus c_2^2 \big(\tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \otimes \mathbf{1} + \mathbf{1} \otimes \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} + a^\dagger_{s,\vec{\kappa}} \otimes a_{s,\vec{\kappa}} + a_{s,\vec{\kappa}} \otimes a^\dagger_{s,\vec{\kappa}}\big) \\
&\oplus c_3^2 \big(\tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \\
&+ a_{s,\vec{\kappa}} \otimes a^\dagger_{s,\vec{\kappa}} \otimes \mathbf{1} + a^\dagger_{s,\vec{\kappa}} \otimes a_{s,\vec{\kappa}} \otimes \mathbf{1} + \mathbf{1} \otimes a_{s,\vec{\kappa}} \otimes a^\dagger_{s,\vec{\kappa}} \\
&+ \mathbf{1} \otimes a^\dagger_{s,\vec{\kappa}} \otimes a_{s,\vec{\kappa}} + a^\dagger_{s,\vec{\kappa}} \otimes \mathbf{1} \otimes a_{s,\vec{\kappa}} + a_{s,\vec{\kappa}} \otimes \mathbf{1} \otimes a^\dagger_{s,\vec{\kappa}}\big) \oplus \cdots \big]
\end{aligned} \tag{66}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator. Comparing this with the generator of time translations

$$\begin{aligned}
\mathcal{H} = c\mathcal{P}_0 = \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} \big[&\tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \oplus \big(\tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \otimes \mathbf{1} + \mathbf{1} \otimes \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\}\big) \\
&\oplus \big(\tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \tfrac{1}{2}\{a_{s,\vec{\kappa}}, a^\dagger_{s,\vec{\kappa}}\}\big) \oplus \cdots \big]
\end{aligned} \tag{67}$$

we can see that there *is* a relation between $\mathcal{H}$ and $\boldsymbol{H}$ but the latter contains terms of the form

$$a_{s,\vec{\kappa}} \otimes a^{\dagger}_{s,\vec{\kappa}} \otimes \mathbf{1} \tag{68}$$

describing interactions between the oscillators (the above term removes energy from the first oscillator and excites the second one by the same amount). The contribution from these interactions vanishes on vacuum states. Below, when we introduce the notion of a generalized coherent state, we will be able to relate averages of $\mathcal{H}$ and $\boldsymbol{H}$. In a similar way one can introduce the 'Pointing operator'

$$\vec{\boldsymbol{P}} = \int_V \mathrm{d}^3x\, \vec{\mathcal{E}}(t,\vec{x}) \times \vec{\mathcal{B}}(t,\vec{x}) = \tfrac{1}{2}\sum_{s,\vec{\kappa}} \hbar\vec{\kappa}\left(\boldsymbol{a}^{\dagger}_{s,\vec{\kappa}}\boldsymbol{a}_{s,\vec{\kappa}} + \boldsymbol{a}_{s,\vec{\kappa}}\boldsymbol{a}^{\dagger}_{s,\vec{\kappa}}\right). \tag{69}$$

Its relation to the generator of 3-translations $\vec{\mathcal{P}}$ is similar to this between $\mathcal{H}$ and $\boldsymbol{H}$.

In the above construction the only element which is beyond a simple transition to many oscillators is the choice of c_n. For different choices of these constants we obtain different representations of non-CCR and therefore also different quantization schemes. Several different ways of reasoning lead to $c_n = 1/\sqrt{n}$ as we shall also see in the next sections.

5. Some particular states

We assume that all the multi-oscillator states are symmetric with respect to permutations of the oscillators.

5.1. Generalized coherent states

For general c_n an eigenstate of $\boldsymbol{a}_{s,\vec{\kappa}}$ corresponding to the eigenvalue $\alpha_{s,\vec{\kappa}}$ is of the form

$$\begin{aligned}|\boldsymbol{\alpha}_{s,\vec{\kappa}}\rangle = {} & f_1(s,\vec{\kappa})|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}/c_1\rangle \oplus f_2(s,\vec{\kappa})|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}/(2c_2)\rangle|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}/(2c_2)\rangle \\ & \oplus f_3(s,\vec{\kappa})|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}/(3c_3)\rangle|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}/(3c_3)\rangle|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}/(3c_3)\rangle \oplus \cdots\end{aligned} \tag{70}$$

where

$$|s,\vec{\kappa},\alpha_{s,\vec{\kappa}}\rangle = |s,\vec{\kappa}\rangle|\alpha_{s,\vec{\kappa}}\rangle \tag{71}$$

$\sum_{n=1}^{\infty}|f_n(s,\vec{\kappa})|^2 = 1$ and $a|\alpha_{s,\vec{\kappa}}\rangle = \alpha_{s,\vec{\kappa}}|\alpha_{s,\vec{\kappa}}\rangle$. What is interesting is that not all f_n have to be non-vanishing.

The average 'energies' of the field in the above eigenstate are

$$\langle\boldsymbol{\alpha}_{s,\vec{\kappa}}|\mathcal{H}|\boldsymbol{\alpha}_{s,\vec{\kappa}}\rangle = \hbar\omega_{\vec{\kappa}}|\alpha_{s,\vec{\kappa}}|^2\sum_{n=1}^{\infty}\frac{1}{nc_n^2}|f_n(s,\vec{\kappa})|^2 + \tfrac{1}{2}\hbar\omega_{\vec{\kappa}}\sum_{n=1}^{\infty}n|f_n(s,\vec{\kappa})|^2 \tag{72}$$

and

$$\langle\boldsymbol{\alpha}_{s,\vec{\kappa}}|\boldsymbol{H}|\boldsymbol{\alpha}_{s,\vec{\kappa}}\rangle = \hbar\omega_{\vec{\kappa}}|\alpha_{s,\vec{\kappa}}|^2 + \tfrac{1}{2}\hbar\omega_{\vec{\kappa}}\sum_{n=1}^{\infty}nc_n^2|f_n(s,\vec{\kappa})|^2. \tag{73}$$

The two averages will differ only by the value of the vacuum contribution if $c_n = 1/\sqrt{n}$ which leads us back to the above-mentioned choice of c_n. With this choice and taking the general combination of coherent states

$$|\boldsymbol{\Psi}\rangle = \sum_{s,\vec{\kappa}}\Phi_{s,\vec{\kappa}}|\boldsymbol{\alpha}_{s,\vec{\kappa}}\rangle \tag{74}$$

we find

$$\langle\Psi|\mathcal{H}|\Psi\rangle = \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2 |\alpha_{s,\vec{\kappa}}|^2 + \tfrac{1}{2} \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2 \sum_{n=1}^{\infty} n |f_n(s,\vec{\kappa})|^2 \tag{75}$$

and

$$\langle\Psi|\boldsymbol{H}|\Psi\rangle = \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2 |\alpha_{s,\vec{\kappa}}|^2 + \tfrac{1}{2} \sum_{s,\vec{\kappa}} \hbar\omega_{\vec{\kappa}} |\Phi_{s,\vec{\kappa}}|^2. \tag{76}$$

One may wonder, then, what is the more natural choice of the free-field Hamiltonian: $\boldsymbol{H}$ describing interacting oscillators, or $\mathcal{H}$ describing the non-interacting ones? The coherent-state average of $\boldsymbol{H}$ does not depend on the average number of oscillators and naturally includes the process of energy exchange between different oscillators. Moreover, both $\boldsymbol{H}$ and $\vec{\boldsymbol{P}}$ are defined in the standard way in terms of the multi-oscillator non-CCR algebra. With this choice of free dynamics we find

$$\mathrm{e}^{\mathrm{i}\boldsymbol{H}t/\hbar} \boldsymbol{a}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\boldsymbol{H}t/\hbar} = \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t \hat{\mathbf{1}}_{s,\vec{\kappa}}} \boldsymbol{a}_{s,\vec{\kappa}} \tag{77}$$

as opposed to the standard formula

$$\mathrm{e}^{\mathrm{i}\mathcal{H}t/\hbar} \boldsymbol{a}_{s,\vec{\kappa}} \mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar} = \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}} t} \boldsymbol{a}_{s,\vec{\kappa}}. \tag{78}$$

The latter choice is simpler because it leads to the standard form of the interaction-picture Hamiltonian and therefore will be the basis of our non-canonical quantum optics. The version based on $\boldsymbol{H}$ is a subject of ongoing study.

5.2. Vacuum

Similarly to the one-oscillator case, the traditional notion of a vacuum state is replaced in our formalism by a vacuum *subspace* consisting of all the vectors annihilated by all $\boldsymbol{a}_{s,\vec{\kappa}}$. Their general form is

$$|\mathbf{0}\rangle = \sum_{s,\vec{\kappa}} O^{(1)}_{s,\vec{\kappa},0} |s,\vec{\kappa},0\rangle \oplus \sum_{s_j,\vec{\kappa}_j} O^{(2)}_{s_1,s_2,\vec{\kappa}_1,\vec{\kappa}_2,0,0} |s_1,\vec{\kappa}_1,0\rangle |s_2,\vec{\kappa}_2,0\rangle$$
$$\oplus \sum_{s_j,\vec{\kappa}_{\lambda_j}} O^{(3)}_{s_1,s_2,s_3\vec{\kappa}_{\lambda_1},\vec{\kappa}_{\lambda_2},\vec{\kappa}_{\lambda_3},0,0,0} |s_1,\vec{\kappa}_1,0\rangle |s_2,\vec{\kappa}_2,0\rangle |s_3,\vec{\kappa}_3,0\rangle \oplus \cdots. \tag{79}$$

It seems that there is no reason for introducing the standard unique 'vacuum state' understood as the cyclic vector of the GNS construction.

As an example of a vacuum state consider

$$|\mathbf{0}\rangle = \sqrt{p_1}|O\rangle \oplus \sqrt{p_2}|O\rangle|O\rangle \oplus \sqrt{p_3}|O\rangle|O\rangle|O\rangle \oplus \cdots. \tag{80}$$

The average energy of the free-field vacuum state is therefore

$$\overline{\mathcal{H}} = \langle\mathbf{0}|\mathcal{H}|\mathbf{0}\rangle = \sum_{n=1}^{\infty} n p_n \langle O|H|O\rangle = \overline{n}\overline{H} \tag{81}$$

where $\overline{n}$ and $\overline{H}$ are, respectively, the average number of oscillators and the average energy of a single oscillator. For the sake of completeness let us note that

$$\overline{\boldsymbol{H}} = \langle\mathbf{0}|\boldsymbol{H}|\mathbf{0}\rangle = \sum_{n=1}^{\infty} c_n^2 n p_n \langle O|H|O\rangle = \overline{H} \sum_{n=1}^{\infty} c_n^2 n p_n. \tag{82}$$

For $c_n = 1$ we obtain $\overline{\boldsymbol{H}} = \overline{\mathcal{H}}$; for $c_n = 1/\sqrt{n}$ $\overline{\boldsymbol{H}} = \overline{H}$, the latter being independent of the number of oscillators. In both cases no problem with infinite vacuum energy is found. Obviously, one can also contemplate other vacua, say, in entangled or mixed states.

The idea that the vacuum may have a non-trivial structure is not new. The Casimir effect, for example, is based on the fact that the structure of frequencies allowed by given boundary conditions may be different on different sides of the boundary. In the non-CCR formalism this corresponds to a modification of the spectrum of Ω. The 'cyclic and unique' vacuum of the GNS construction is only historically related to the idea of Heisenberg and others that the field is a collection of quantum harmonic oscillators. The modifications of the vacuum observed in measurements of the Casimir force are not, in the standard approach, related to modifications of the vacuum state. The non-canonical formulation suggests that both Ω and the vacuum *state* should be modified as a consequence of the Casimir boundary conditions. One may expect that the force may be different for different vacuum states. The problem requires further studies.

5.3. Multi-photon states

Assume $|\mathbf{0}\rangle$ is *a* vacuum state. The non-canonical algebra (non-CCR) differs from the canonical one in the commutator

$$[\boldsymbol{a}_\lambda, \boldsymbol{a}_\lambda^\dagger] = \hat{\mathbf{1}}_\lambda \tag{83}$$

where for any λ, λ'

$$[\hat{\mathbf{1}}_{\lambda'}, \boldsymbol{a}_\lambda] = [\hat{\mathbf{1}}_{\lambda'}, \boldsymbol{a}_\lambda^\dagger] = 0 \tag{84}$$

where λ denotes $(s, \vec{\kappa})$. A normalized state describing a collection of photons is defined in analogy to the standard formalism by

$$\frac{1}{\sqrt{n_1!n_2!\ldots n_N!\langle\mathbf{0}|\hat{\mathbf{1}}_{\lambda_N}^{n_N}\ldots\hat{\mathbf{1}}_{\lambda_2}^{n_2}\hat{\mathbf{1}}_{\lambda_1}^{n_1}|\mathbf{0}\rangle}}(\boldsymbol{a}_{\lambda_N}^\dagger)^{n_N}\ldots(\boldsymbol{a}_{\lambda_2}^\dagger)^{n_2}(\boldsymbol{a}_{\lambda_1}^\dagger)^{n_1}|\mathbf{0}\rangle =: |\boldsymbol{n}_{\lambda_1}, \boldsymbol{n}_{\lambda_2}, \ldots, \boldsymbol{n}_{\lambda_N}\rangle. \tag{85}$$

Physically the state represents excitations of the ensemble of harmonic oscillators. As such it depends on the statistics of the ensemble, that is the concrete choice of $|\mathbf{0}\rangle$. States corresponding to the same λ but different ns, or to the same n but different λs, are orthogonal. Consequently, a non-canonical vacuum average of any product of non-canonical creation and annihilation operators vanishes if and only if an analogous expression formulated in terms of the canonical objects does. This property is a consequence of three facts which hold true in both formalisms: (a) annihilation operators annihilate vacuum states, (b) creation operators are obtained by Hermitian conjugation from the annihilation operators and (c) the right-hand side of a commutator of creation and annihilation operators commutes with all creation and annihilation operators.

6. Perturbation theory

It is essential that, similarly to the one-oscillator formalism, the free Hamiltonian (defined simply as a generator of time translations) generates the standard dynamics of annihilation operators:

$$\mathrm{e}^{\mathrm{i}\mathcal{H}t/\hbar}\boldsymbol{a}_{s,\vec{\kappa}}\mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar} = \mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}}t}\boldsymbol{a}_{s,\vec{\kappa}}. \tag{86}$$

Accordingly, the form of the interaction-picture Hamiltonian will by the same as in the standard theory. This would not have been quite the same if we had chosen $\boldsymbol{H}$ in the role of the free

Hamiltonian (an option which, nevertheless, should be investigated). In what follows we start with $H = H_0 + V$, where

$$H_0 = H_A + \mathcal{H} \tag{87}$$

$$V = -\frac{e}{m}\vec{\mathcal{A}}(\vec{x})\cdot\vec{p}$$

$$= -\frac{e}{m}\sum_{s,\vec{\kappa}}\sqrt{\frac{\hbar}{2\omega_{\vec{\kappa}}V}}\left(\boldsymbol{a}_{s,\vec{\kappa}}\mathrm{e}^{\mathrm{i}\vec{\kappa}\cdot\vec{x}}\vec{e}_{s,\vec{\kappa}}\cdot\vec{p} + \boldsymbol{a}^{\dagger}_{s,\vec{\kappa}}\mathrm{e}^{-\mathrm{i}\vec{\kappa}\cdot\vec{x}}\vec{e}^{\,*}_{s,\vec{\kappa}}\cdot\vec{p}\right). \tag{88}$$

In the interaction picture we obtain

$$V(t) = -\frac{e}{m}\vec{\mathcal{A}}(t,\vec{x})\cdot\vec{p}(t)$$

$$= -\frac{e}{m}\sum_{s,\vec{\kappa}}\sqrt{\frac{\hbar}{2\omega_{\vec{\kappa}}V}}\left(\boldsymbol{a}_{s,\vec{\kappa}}\mathrm{e}^{-\mathrm{i}(\omega_{\vec{\kappa}}t-\vec{\kappa}\cdot\vec{x})}\vec{e}_{s,\vec{\kappa}}\cdot\vec{p}(t) + \boldsymbol{a}^{\dagger}_{s,\vec{\kappa}}\mathrm{e}^{\mathrm{i}(\omega_{\vec{\kappa}}t-\vec{\kappa}\cdot\vec{x})}\vec{e}^{\,*}_{s,\vec{\kappa}}\cdot\vec{p}(t)\right) \tag{89}$$

and

$$\vec{p}(t) = \mathrm{e}^{\mathrm{i}H_A t}\,\vec{p}\,\mathrm{e}^{-\mathrm{i}H_A t}. \tag{90}$$

Since we are purposefully neglecting the '$\vec{A}^{\,2}$' term in the Hamiltonian, one should restrict the analysis to the dipole approximation and therefore it is justified to set $\vec{x} = 0$:

$$V(t) = \sum_{\lambda}\left(\boldsymbol{a}_{\lambda}\hat{g}_{\lambda}(t) + \boldsymbol{a}^{\dagger}_{\lambda}\hat{g}^{\dagger}_{\lambda}(t)\right). \tag{91}$$

The operators

$$\hat{g}_{s,\vec{\kappa}}(t) = -\frac{e}{m}\sqrt{\frac{\hbar}{2\omega_{\vec{\kappa}}V}}\mathrm{e}^{-\mathrm{i}\omega_{\vec{\kappa}}t}\vec{e}_{s,\vec{\kappa}}\cdot\vec{p}(t) = \hat{g}_{\lambda}(t) \tag{92}$$

are identical to those from the standard formalism and act only on atomic degrees of freedom (i.e. commute with $\boldsymbol{a}_{s,\vec{\kappa}}$).

6.1. Spontaneous decay of an excited state

The first problem we shall treat in a non-canonical way is the lifetime of an excited atomic state. The problem, as we shall see, is of particular importance for the physical interpretation of the non-canonical formalism.

Assume that at $t = 0$ the atom–field system is described by the state $|\Psi(0)\rangle = |\mathbf{0}, A\rangle$. The amplitude that the atom remains in the excited state is

$$\langle\mathbf{0}, A|\Psi(t)\rangle = 1 + \frac{1}{(\mathrm{i}\hbar)^2}\int_0^t \mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_1\sum_{\lambda_1\lambda_2}\langle\mathbf{0}|\boldsymbol{a}_{\lambda_2}\boldsymbol{a}^{\dagger}_{\lambda_1}|\mathbf{0}\rangle\langle A|\hat{g}_{\lambda_2}(t_2)\hat{g}^{\dagger}_{\lambda_1}(t_1)|A\rangle$$

$$+\frac{1}{(\mathrm{i}\hbar)^4}\int_0^t \mathrm{d}t_4\int_0^{t_4}\mathrm{d}t_3\int_0^{t_3}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_1$$

$$\times\sum_{\lambda_1\lambda_2\lambda_3\lambda_4}\langle\mathbf{0}|\boldsymbol{a}_{\lambda_4}\boldsymbol{a}^{\dagger}_{\lambda_3}\boldsymbol{a}_{\lambda_2}\boldsymbol{a}^{\dagger}_{\lambda_1}|\mathbf{0}\rangle\langle A|\hat{g}_{\lambda_4}(t_4)\hat{g}^{\dagger}_{\lambda_3}(t_3)\hat{g}_{\lambda_2}(t_2)\hat{g}^{\dagger}_{\lambda_1}(t_1)|A\rangle$$

$$+\frac{1}{(\mathrm{i}\hbar)^4}\int_0^t \mathrm{d}t_4\int_0^{t_4}\mathrm{d}t_3\int_0^{t_3}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_1$$

$$\times \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \langle \mathbf{0}|\boldsymbol{a}_{\lambda_4}\boldsymbol{a}_{\lambda_3}\boldsymbol{a}^{\dagger}_{\lambda_2}\boldsymbol{a}^{\dagger}_{\lambda_1}|\mathbf{0}\rangle\langle A|\hat{g}_{\lambda_4}(t_4)\hat{g}_{\lambda_3}(t_3)\hat{g}^{\dagger}_{\lambda_2}(t_2)\hat{g}^{\dagger}_{\lambda_1}(t_1)|A\rangle + \cdots$$

$$= 1 + \frac{1}{(\mathrm{i}\hbar)^2}\int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \sum_{\lambda_1\lambda_2} \langle\hat{0}|\hat{a}_{\lambda_2}\hat{a}^{\dagger}_{\lambda_1}|\hat{0}\rangle\langle A|\hat{g}_{\lambda_2}(t_2)\hat{g}^{\dagger}_{\lambda_1}(t_1)|A\rangle X^{01}_{\lambda_2\lambda_1}$$

$$+\frac{1}{(\mathrm{i}\hbar)^4}\int_0^t \mathrm{d}t_4 \int_0^{t_4} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1$$

$$\times \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \langle\hat{0}|\hat{a}_{\lambda_4}\hat{a}^{\dagger}_{\lambda_3}\hat{a}_{\lambda_2}\hat{a}^{\dagger}_{\lambda_1}|\hat{0}\rangle\langle A|\hat{g}_{\lambda_4}(t_4)\hat{g}^{\dagger}_{\lambda_3}(t_3)\hat{g}_{\lambda_2}(t_2)\hat{g}^{\dagger}_{\lambda_1}(t_1)|A\rangle X^{0101}_{\lambda_4\lambda_3\lambda_2\lambda_1}$$

$$+\frac{1}{(\mathrm{i}\hbar)^4}\int_0^t \mathrm{d}t_4 \int_0^{t_4} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1$$

$$\times \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \langle\hat{0}|\hat{a}_{\lambda_4}\hat{a}_{\lambda_3}\hat{a}^{\dagger}_{\lambda_2}\hat{a}^{\dagger}_{\lambda_1}|\hat{0}\rangle\langle A|\hat{g}_{\lambda_4}(t_4)\hat{g}_{\lambda_3}(t_3)\hat{g}^{\dagger}_{\lambda_2}(t_2)\hat{g}^{\dagger}_{\lambda_1}(t_1)|A\rangle X^{0011}_{\lambda_4\lambda_3\lambda_2\lambda_1} + \cdots. \tag{93}$$

In the above perturbative expansion we have explicitly shown all the non-vanishing terms up to the fifth order of perturbation theory. Here $\langle\hat{0}|\hat{a}_{\lambda_4}\hat{a}_{\lambda_3}\hat{a}^{\dagger}_{\lambda_2}\hat{a}^{\dagger}_{\lambda_1}|\hat{0}\rangle$, etc are the expressions one would have obtained in ordinary canonical formalism and

$$X^{0011}_{\lambda_4\lambda_3\lambda_2\lambda_1} = \frac{\langle \mathbf{0}|\boldsymbol{a}_{\lambda_4}\boldsymbol{a}_{\lambda_3}\boldsymbol{a}^{\dagger}_{\lambda_2}\boldsymbol{a}^{\dagger}_{\lambda_1}|\mathbf{0}\rangle}{\langle\hat{0}|\hat{a}_{\lambda_4}\hat{a}_{\lambda_3}\hat{a}^{\dagger}_{\lambda_2}\hat{a}^{\dagger}_{\lambda_1}|\hat{0}\rangle} \tag{94}$$

and so on. Such expressions are well defined since whenever their denominators vanish the whole term it containing vanishes as well. This fact is of crucial importance and shows that the perturbative expansions in both canonical and non-canonical frameworks contain terms of *exactly the same type* but differing by the numerical factors $X^{\cdots}_{\cdots}$.

6.2. Remark on the choice of algebras and their representations

Let us note that in the above calculation we have not used the explicit realization of the non-canonical algebra but only the algebra itself. At such a general level both the canonical and non-canonical theories can be regarded as particular cases of a more general theory characterized by the algebra

$$[a_\lambda, a^{\dagger}_{\lambda'}] = \delta_{\lambda\lambda'} I_\lambda \tag{95}$$

$$[a_\lambda, a_{\lambda'}] = 0 \tag{96}$$

$$[a^{\dagger}_\lambda, a^{\dagger}_{\lambda'}] = 0 \tag{97}$$

$$[a^{\dagger}_\lambda, I_{\lambda'}] = 0 \tag{98}$$

$$[a_\lambda, I_{\lambda'}] = 0 \tag{99}$$

$$[I_\lambda, I_{\lambda'}] = 0. \tag{100}$$

The canonical choice, based on oscillators with the classical parameter λ, is $I_\lambda = \mathbf{1}$; the choice based on oscillators with eigenvalue λ is $I_\lambda = \hat{\mathbf{1}}_\lambda$.

Of particular interest in this context turns out to be the formula (56). Replacing 1_λ by $\mathbf{1}$ and taking $c_n = 1/\sqrt{n}$ we again obtain a representation of the *canonical* commutation relations.

However, this representation is *not* the Fock one since acting with a creation operator on a vacuum state we obtain a single-photon state which has non-vanishing components in all the multi-oscillator subspaces. The general quantization scheme we are discussing may be regarded as containing the standard canonical theory but in a non-standard representation. The existence of such a possibility is not so surprising if one recalls the classic result due to von Neumann, stating that systems with infinite degrees of freedom admit uncountably many inequivalent representations of CCR [10].

6.3. Example: concrete choice of vacuum and I_λ

To proceed further and obtain more information as to the physical meaning of the non-canonical dynamics we have to make the analysis less general. First of all let us stick to the particular choice of $\hat{\mathbf{1}}_\lambda$ in terms of POV measures we have introduced earlier and assume that $\sum_\lambda \hat{\mathbf{1}}_\lambda = \mathbf{1}$. Second, let us take the vacuum state in the form (80). Under such assumptions we can explicitly compute the factors $X^{\cdots}_{\cdots}$:

$$X^{01}_{\lambda\lambda} = \frac{\langle \mathbf{0}|\boldsymbol{a}_\lambda \boldsymbol{a}^\dagger_\lambda|\mathbf{0}\rangle}{\langle \hat{0}|\hat{a}_\lambda \hat{a}^\dagger_\lambda|\hat{0}\rangle} = \langle \mathbf{0}|\hat{\mathbf{1}}_\lambda|\mathbf{0}\rangle = |O_\lambda|^2 \tag{101}$$

$$X^{0101}_{\lambda\lambda\lambda'\lambda'} = \frac{\langle \mathbf{0}|\boldsymbol{a}_\lambda \boldsymbol{a}^\dagger_\lambda \boldsymbol{a}_{\lambda'} \boldsymbol{a}^\dagger_{\lambda'}|\mathbf{0}\rangle}{\langle \hat{0}|\hat{a}_\lambda \hat{a}^\dagger_\lambda \hat{a}_{\lambda'} \hat{a}^\dagger_{\lambda'}|\hat{0}\rangle} = \langle \mathbf{0}|\hat{\mathbf{1}}_\lambda \hat{\mathbf{1}}_{\lambda'}|\mathbf{0}\rangle = \sum_{n=1}^{\infty} p_n \left(1 - \frac{1}{n}\right)|O_\lambda|^2 |O_{\lambda'}|^2$$

$$= \big(1 - \langle 1/n\rangle\big)|O_\lambda|^2 |O_{\lambda'}|^2 \tag{102}$$

$$X^{0101}_{\lambda\lambda\lambda\lambda} = \frac{\langle \mathbf{0}|\boldsymbol{a}_\lambda \boldsymbol{a}^\dagger_\lambda \boldsymbol{a}_\lambda \boldsymbol{a}^\dagger_\lambda|\mathbf{0}\rangle}{\langle \hat{0}|\hat{a}_\lambda \hat{a}^\dagger_\lambda \hat{a}_\lambda \hat{a}^\dagger_\lambda|\hat{0}\rangle} = \langle \mathbf{0}|\hat{\mathbf{1}}^2_\lambda|\mathbf{0}\rangle = \big(1 - \langle 1/n\rangle\big)|O_\lambda|^4 + \langle 1/n\rangle |O_\lambda|^2 \tag{103}$$

$$X^{0011}_{\lambda\lambda\lambda\lambda} = \frac{\langle \mathbf{0}|\boldsymbol{a}_\lambda \boldsymbol{a}_\lambda \boldsymbol{a}^\dagger_\lambda \boldsymbol{a}^\dagger_\lambda|\mathbf{0}\rangle}{\langle \hat{0}|\hat{a}_\lambda \hat{a}_\lambda \hat{a}^\dagger_\lambda \hat{a}^\dagger_\lambda|\hat{0}\rangle} = \langle \mathbf{0}|\hat{\mathbf{1}}^2_\lambda|\mathbf{0}\rangle = \big(1 - \langle 1/n\rangle\big)|O_\lambda|^4 + \langle 1/n\rangle |O_\lambda|^2 \tag{104}$$

$$X^{0011}_{\lambda\lambda'\lambda\lambda'} = \frac{\langle \mathbf{0}|\boldsymbol{a}_\lambda \boldsymbol{a}_{\lambda'} \boldsymbol{a}^\dagger_\lambda \boldsymbol{a}^\dagger_{\lambda'}|\mathbf{0}\rangle}{\langle \hat{0}|\hat{a}_\lambda \hat{a}_{\lambda'} \hat{a}^\dagger_\lambda \hat{a}^\dagger_{\lambda'}|\hat{0}\rangle} = \langle \mathbf{0}|\hat{\mathbf{1}}_\lambda \hat{\mathbf{1}}_{\lambda'}|\mathbf{0}\rangle = \big(1 - \langle 1/n\rangle\big)|O_\lambda|^2 |O_{\lambda'}|^2. \tag{105}$$

What is interesting (and very characteristic) is that all these factors are smaller than 1 (this follows trivially from $\sum_\lambda |O_\lambda|^2 = 1$). An analysis of higher-order terms shows that this is a generic property of the non-canonical perturbation theory. The n occurring in the average $\langle 1/n\rangle$ is the number-of-oscillators operator. For realistic vacua one may expect the average number of oscillators to be large and therefore $\langle 1/n\rangle \approx 0$. Taking a more general vacuum state we arrive at $\langle 1/n_\lambda\rangle$ instead of $\langle 1/n\rangle$, which means that the influence of the vacuum may vary from frequency to frequency (i.e. from point to point in space).

Now, if we ignore the corrections coming from $\langle 1/n\rangle$ we can see that the non-canonical perturbative expansion of the amplitude is the same as we would have obtained by using the *standard theory* but with $\hat{g}_\lambda$ regularized by $\hat{g}_\lambda \to O_\lambda \hat{g}_\lambda$. Consequently, there exists a natural cut-off in the theory which follows *only* from the fact that the vacuum wavefunction is square-integrable and therefore $O_{s,\vec{\kappa}} \to 0$ for $|\vec{\kappa}| \to \infty$. It is quite remarkable that the same mechanism that eliminated the infinite vacuum energy plays a similar role in the other parts of the theory. As we shall see shortly the actual role of the vacuum can be analysed only in a fully relativistic setting since then the charge and mass renormalization come into play.

However, for the sake of concreteness and to make some rough estimates of the effects involved let us take the trivial example where the vacuum amplitudes are constant, say, $O_{s,\vec{\kappa}} = C$ for all $\omega_{\vec{\kappa}} < \omega_{\max}$ and zero otherwise. The dynamics of the amplitude is then

(up to $\langle 1/n\rangle \approx 0$) equivalent to the standard one with the cut-off at $\omega_{\max}$ and with the coupling constant e/m replaced by Ce/m. This implies that e and m have to be treated as *bare* parameters and the experimental value is $e_{\rm ex}/m_{\rm ex} = Ce/m$. With this observation in mind we can discuss non-canonically other perturbative effects which are widely believed to be a consequence of the standard canonical quantization. Below we find it useful to make the bare parameter e/m explicit in calculations and for this reason will use the notation $\hat{g}_\lambda = (e/m)\hat{f}_\lambda$.

6.4. Spontaneous emission of N identical photons in Nth-order perturbation theory

Assume the atomic transition is $|A\rangle \to |B\rangle$. Up to the Nth-order perturbative correction we obtain

$$\begin{aligned}\langle \boldsymbol{N}_\lambda, B|\Psi(t)\rangle &= \left(\frac{e}{m}\right)^N \frac{1}{(\mathrm{i}\hbar)^N}\int_0^t \mathrm{d}t_N \int_0^{t_N} \ldots \int_0^{t_2} \mathrm{d}t_1 \langle \boldsymbol{N}_\lambda|\boldsymbol{a}_\lambda^{\dagger N}|\mathbf{0}\rangle\langle B|\hat{f}_\lambda^\dagger(t_N)\ldots\hat{f}_\lambda^\dagger(t_1)|A\rangle \\ &= \left(\frac{e}{m}\right)^N \frac{\langle \boldsymbol{N}_\lambda|\boldsymbol{a}_\lambda^{\dagger N}|\mathbf{0}\rangle}{\langle \hat{N}_\lambda|\hat{a}_\lambda^{\dagger N}|\hat{0}\rangle} \times (\text{relevant canonical formula}) \\ &= \left(\frac{e}{m}\right)^N \sqrt{\langle \mathbf{0}|\hat{\mathbf{1}}_\lambda^N|\mathbf{0}\rangle} \times (\text{relevant canonical formula}) \end{aligned} \tag{106}$$

where the 'hatted' expressions are those from the canonical theory. As we can see the task is reduced to computing $\langle \mathbf{0}|\hat{\mathbf{1}}_\lambda^N|\mathbf{0}\rangle$. The general formula, valid for any N, is somewhat complicated and not very illuminating. The cases $N = 1$ and 2 we have already met. Making the simplifying choice of a very 'flat' distribution of the vacuum modes (i.e. $O_{s,\vec{\kappa}} = C$ below some threshold) we obtain

$$\langle \mathbf{1}_\lambda, B|\Psi(t)\rangle = C\langle \hat{1}_\lambda, B|\hat{\Psi}(t)\rangle \tag{107}$$

$$\langle \mathbf{2}_\lambda, B|\Psi(t)\rangle = C^2\sqrt{1 - \langle 1/n\rangle + \langle 1/n\rangle C^{-2}}\,\langle \hat{2}_\lambda, B|\hat{\Psi}(t)\rangle. \tag{108}$$

Identifying Ce/m with $e_{\rm ex}/m_{\rm ex}$ we obtain the standard quantum-optics results provided

$$\sqrt{1 - \langle 1/n\rangle + \langle 1/n\rangle C^{-2}} \approx 1. \tag{109}$$

The assumption we have made concerning the form of the vacuum state means that $C^{-2} = n_{\rm s}$ is the number of states whose probability is non-zero. The proportionality factor is therefore

$$\sqrt{1 - \langle 1/n\rangle + \langle n_{\rm s}/n\rangle}. \tag{110}$$

We can proceed further, more in the spirit of the canonical theory, and assume that each oscillator has a concrete frequency, for each different frequency we have a separate oscillator, and each oscillator has two equally probable polarizations. Then $\langle n_{\rm s}/n\rangle \approx 2$. The ratio of probabilities of one- and two-photon spontaneous emissions obtained in the non-canonical theory becomes, approximately,

$$(P_1/P_2)_{\text{non-CCR}} \approx \tfrac{1}{3}(P_1/P_2)_{\rm CCR} \tag{111}$$

where $(P_1/P_2)_{\rm CCR}$ denotes the standard canonical result obtained with a cut-off at $\omega = \omega_{\max}$. Although the above approximations are very crude the result shows that precise measurements of the ratio may provide estimates on the proportionality factor and, hence, indirectly test the non-canonical theory.

For $N = 3$,

$$\langle \mathbf{0}|\hat{\mathbf{1}}_\lambda^3|\mathbf{0}\rangle = \sum_{n=1}^{\infty} \frac{1}{n^3}\big(nC^2 + 3n(n-1)C^4 + (n^3 - 3n^2 + 2n)C^6\big)p_n$$
$$= C^6 + 3C^4\big(1 - C^2\big)\big\langle 1/n\big\rangle + C^2\big(1 - 3C^2 + 2C^4\big)\big\langle 1/n^2\big\rangle. \tag{112}$$

As before the result becomes the standard one if the approximation

$$\sqrt{C^6 + 3C^4\big(1 - C^2\big)\big\langle 1/n\big\rangle + C^2\big(1 - 3C^2 + 2C^4\big)\big\langle 1/n^2\big\rangle} \approx C^3 \tag{113}$$

is justified. Let us note that in standard canonical quantum optics one considers a vacuum consisting of an *infinite* number of oscillators and such subtleties are trivially ignored.

6.5. Spontaneous emission of two different photons

By the same argument as before the proportionality factor we need to estimate in second-order perturbation theory is

$$\frac{\langle \mathbf{1}_\lambda, \mathbf{1}_{\lambda'}|\boldsymbol{a}_\lambda^\dagger \boldsymbol{a}_{\lambda'}^\dagger|\mathbf{0}\rangle}{\langle \hat{1}_\lambda, \hat{1}_{\lambda'}|\hat{a}_\lambda^\dagger \hat{a}_{\lambda'}^\dagger|\hat{0}\rangle} = \sqrt{\langle \mathbf{0}|\hat{\mathbf{1}}_\lambda \hat{\mathbf{1}}_{\lambda'}|\mathbf{0}\rangle} = \sqrt{1 - \big\langle 1/n\big\rangle}C^2 \approx C^2 \tag{114}$$

so that the result essentially agrees with the canonical one. It is also in perfect agreement with the explicit calculations given in [11].

6.6. Stimulated emission

The final example we will discuss is the first-order calculation of the transition amplitude $|\boldsymbol{N}_\lambda, A\rangle \to |(\boldsymbol{N} + \mathbf{1})_\lambda, B\rangle$. The appropriate proportionality coefficient is

$$\sqrt{\frac{\langle \mathbf{0}|\hat{\mathbf{1}}_\lambda^{N+1}|\mathbf{0}\rangle}{\langle \mathbf{0}|\hat{\mathbf{1}}_\lambda^{N}|\mathbf{0}\rangle}} \approx C \tag{115}$$

provided the values of $\big\langle 1/n\big\rangle$, $\big\langle 1/n^2\big\rangle$, etc are sufficiently small. This is the correct result since C becomes absorbed into the renormalized coupling constant.

7. Black-body radiation

The final test of the new formalism we want to perform is the problem of black-body radiation. Planck's famous formula [13]

$$\varrho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\mathrm{e}^{\beta\hbar\omega} - 1} = \frac{\hbar}{\pi^2 c^3}\omega^3 \overline{N}_\omega \tag{116}$$

where $\overline{N}_\omega$ is the average number of excitations of an oscillator in inverse temperature β, is one of the first great successes of quantum radiation theory and marks the beginning of quantum mechanics. Contemporary measurements of $\varrho(\omega)$ [14, 15] performed by means of COBE (cosmic background explorer) are in a very good agreement with Planck's law. The data have been carefully analysed in the context of non-extensive statistics [16, 17] in search of possible deviations from extensivity. The result that comes out systematically is $|q - 1| < 10^{-4}$ where q is the Tsallis parameter. The case $q = 1$ corresponds to the exact Planck formula. If there are any corrections whatever, they must be quite small.

The derivation of the formula given by Einstein is based on the properties of spontaneous and stimulated emissions. As we have seen above differences may occur with respect to the standard formalism but under reasonable assumptions they may be expected to be small.

Below we follow another standard route which consists basically of two steps. First, one counts the number of different wavevectors $\vec{k}$ such that $c|\vec{k}| \in [\omega, \omega + \Delta\omega]$. Second, one associates with each such vector an oscillator and counts the average number of its excitations assuming the Boltzmann–Gibbs probability distribution at temperature T and chemical potential $\mu = 0$. The latter assumption is justified by the fact that the number of excitations of the electromagnetic field is not conserved in atom–light interactions.

In the new model the situation is slightly different since there exists an additional conserved quantum number: the number of *oscillators*. As we have seen in previous calculations the Hamiltonian is block-diagonal with respect to $\oplus$ but changes the number of excitations in each N-oscillator subspace of the direct sum. The state vectors at the multi-oscillator level are symmetric with respect to permutations of the oscillators and therefore the oscillators themselves have to be regarded as bosons whose number is conserved and their chemical potential is $\mu \neq 0$. However, their excitations should be regarded as bosons with a vanishing chemical potential.

The eigenvalues of $\mathcal{H}$

$$E_{m,n} = m\hbar\omega\left(n + \tfrac{1}{2}\right) \tag{117}$$

corresponding to the oscillator whose frequency is ω are parametrized by two natural numbers: m (the number of oscillators) and n (the number of excitations). Assuming the standard Boltzmann–Gibbs statistics we obtain the probabilities

$$p_{m,n} = Z^{-1}\mathrm{e}^{-\beta[m\hbar\omega(n+\frac{1}{2})-m\mu]} \tag{118}$$

where

$$Z = \sum_{m=1}^{\infty} \mathrm{e}^{\beta m(\mu+\hbar\omega/2)} \frac{\mathrm{e}^{-\beta m\hbar\omega}}{1-\mathrm{e}^{-\beta m\hbar\omega}}. \tag{119}$$

The Lambert series [18]

$$\sum_{m=1}^{\infty} a_m \frac{x^m}{1-x^m} \tag{120}$$

is convergent for any x if $\sum_{m=1}^{\infty} a_m$ is convergent. Otherwise (120) converges for exactly those x for which the power series $\sum_{m=1}^{\infty} a_m x^m$ does. In (119) $a_m = \mathrm{e}^{\beta m(\mu+\hbar\omega/2)}$ and $\sum_{m=1}^{\infty} a_m < \infty$ if $\mu + \hbar\omega/2 < 0$. If $\mu + \hbar\omega/2 \geqslant 0$ we still have convergence of (119) as long as $\sum_{m=1}^{\infty} \mathrm{e}^{-\beta m[\frac{1}{2}\hbar\omega-\mu]} < \infty$. The upper limit imposed on μ by the finiteness of Z is therefore $\mu < \frac{1}{2}\hbar\omega$. In what follows we assume that μ is ω-independent and therefore $\mu \leqslant 0$.

The appropriate average number of excitations is

$$\overline{n}_\omega = Z^{-1} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} mn\mathrm{e}^{-\beta[m\hbar\omega(n+\frac{1}{2})-m\mu]} \tag{121}$$

and the Planck formula is replaced by

$$\varrho_{\mathrm{new}}(\omega) = \frac{\hbar}{\pi^2 c^3}\omega^3 \overline{n}_\omega. \tag{122}$$

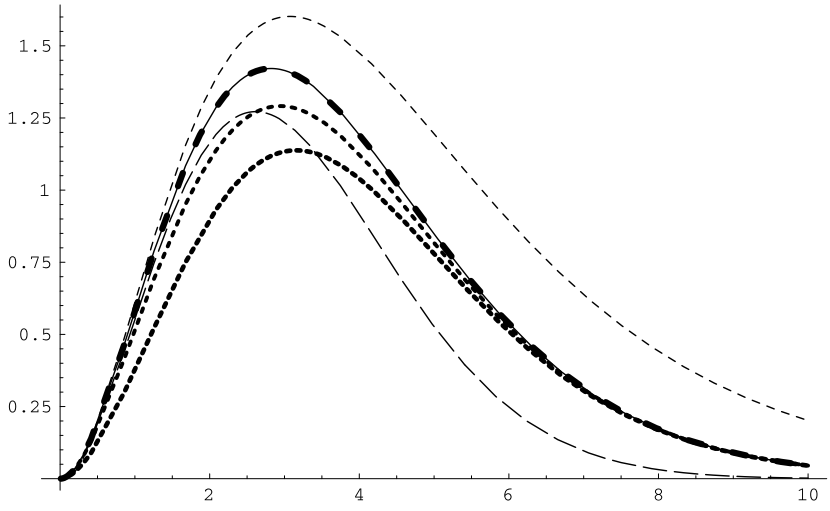

Figure 1.

It is easy to show that $\varrho_{\rm new}(\omega)$ tends to the Planck distribution with $\mu \to -\infty$. To see this consider a more general series

$$Z^{-1}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} mnq_m \mathrm{e}^{-\beta m\hbar\omega(n+\frac{1}{2})} \tag{123}$$

where Z is the normalization factor and $\sum_{m=1}^{\infty} q_m < \infty$. If $q_1 = 1$ and $q_m = 0$ for $m > 1$ then (123) is just the exact Planckian formula. Factoring out $\mathrm{e}^{-\beta|\mu|}$ in both the numerator and the denominator of $\overline{n}_\omega$ we obtain $q_1 = 1$ and $q_m = \mathrm{e}^{-\beta|\mu|(m-1)}$ for $m > 1$. For $|\mu| \to \infty$ all q_m, for $m > 1$, vanish and the limiting distribution is Planckian.

This proves that an experimental agreement with the ordinary Planck's $\varrho(\omega)$ cannot rule out our modification but can, at most, set a lower bound on an admissible value of $|\mu|$. However, assuming that μ has some finite and fixed value it should be, in principle, measurable.

Figure 1 shows the plots of $\varrho_{\rm new}(\omega)$ for $\mu = 0$ (lower dotted), $\mu = -0.8k_{\rm B}T$ (upper dotted) and $\mu = -10k_{\rm B}T$ (full). The thick broken curve is the Planck distribution. The curve obtained for $\mu = -10k_{\rm B}T$ is indistinguishable from the Planck distribution. The plot does not change if one takes $\mu < -10k_{\rm B}T$ and differences are not visible even if one plots the distributions on a logarithmic scale (not shown here). This is a numerical proof that the distribution we have obtained on the basis of the modified quantization tends *very quickly* to the Planck one as $|\mu|$ increases. It is instructive to compare the modification we have predicted with those arising from non-extensive statistics. The two thin broken curves represent Tsallis distributions resulting from a non-extensive formalism for $q = 0.95$ (lower) and $q = 1.05$ (upper). The modifications we have derived are therefore qualitatively different from those resulting from Tsallis statistics.

Assuming that the chemical potential is temperature independent, say $\mu = -k_{\rm B}T_0$, we obtain a kind of critical temperature $T_{\rm critical} \approx T_0/3$ above which the ratio $\mu/(k_{\rm B}T)$ is small enough to make the modifications of the distribution observable. For $T < T_{\rm critical}$ the distribution should be given by the Planck law; for $T > T_{\rm critical}$ the distribution should approach the $\mu = 0$ distribution, i.e. this would be a Planck-type curve but with the maximum lowered and shifted towards higher energies. In figure 2 $\varrho_{\rm new}(\omega)$ is shown for $-10 \leqslant -T_0/T \leqslant 0$.

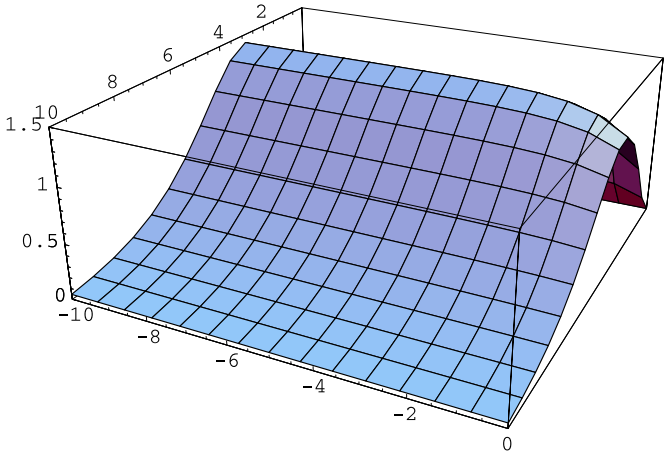

Figure 2.

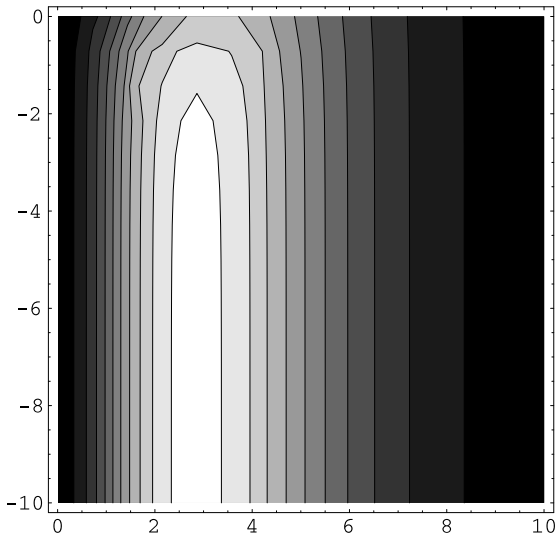

Figure 3.

The cut through $T_0/T = 10$ is practically indistinguishable from the Planck distribution. Figure 3 (the contour plot of figure 2) shows that the modifications become visible around $\mu \approx -3k_{\mathrm{B}}T$.

8. Overview of the physical structure of the construction

The intuitive picture behind the construction is the following.

We begin with the idea that an electromagnetic field is a collection of harmonic oscillators. In the original formulation proposed by Heisenberg, Born, Jordan and Dirac one needed a separate oscillator for each point of space or, which is equivalent, for each 4-momentum.

Therefore, the ensemble they considered was a classical ensemble of an infinite number of quantum oscillators. The energy of such an ensemble is infinite.

Our starting point is based on intuitions originating from modern quantum interferometry (cf [19]). One knows that ensembles of atoms or molecules can be represented by a single-particle state vector even in the case where the spatial extension of the ensemble is macroscopic (say, of the size of the interferometer). It is also known that a coherent ensemble of molecules in the interferometer can interact with external systems (a laser beam) without an irreversible loss of coherence as demonstrated by the decoherence–recoherence experiments of the MIT group [19]. A molecule in an interferometer can be regarded as an example of a harmonic oscillator with indefinite frequency because the momentum-space wavefunction represents a superposition of different velocities of the molecule's centre of mass. These different velocities imply a coherent distribution of Doppler shifts and hence also of frequencies. This observation is the mathematical basis of the technique of laser cooling. There is clear experimental evidence showing that a laser-cooled ensemble of bosons involving a large number of particles (say, 1000) may behave in a coherent way [20].

The main difference between such coherent ensembles and classical ones is that one can still regard the oscillators as 'being at all points of space' even in the case where the number of oscillators is finite. The oscillators forming an electromagnetic field are assumed to be of this kind. We also assume that the number of oscillators may be indefinite. This leads us to a kind of Fock-space formalism but the space does not contain the unique and cyclic 'vacuum state' typical of axiomatic quantum field theory. However, if one really needs such an object for some formal reasons, one is allowed to introduce it, for example by adding to our Hilbert space a one-dimensional Hilbert space consisting of complex phase factors. We do not do this since anyway such a formal 'vacuum state' is physically completely unrelated to the state representing the ensemble of ground-state oscillators, and this is what we think of as a vacuum.

Similarly to the ensemble of oscillators in an interferometer the probability density of their spatial (or momentum) localization does not have to be constant. Consequently, the structure of the vacuum state may vary from point to point and it would be unphysical to insist on only one unique vacuum state.

Once we have the (vacuum) ensemble of oscillators we can contemplate their excitations. For example, if we know that only one oscillator has been excited we obtain a superposed (actually, entangled) state of all the oscillators. In the one-oscillator subspace we have one excited oscillator. In the two-oscillator subspace we have a linear, symmetrized combination corresponding to either the first oscillator excited and the second one in its ground state, or the other way around. A similar situation occurs in each n-oscillator subspace of the direct-sum Hilbert space of the indefinite-number-of-oscillators ensemble. Consequently, the algebraic structure of a one-photon state is different from the usual canonical one and even a single-photon state may have non-vanishing components in all the n-oscillator subspaces. How many such components are present depends on the choice of vacuum. If the vacuum consists of exactly n oscillators, the one-photon state will correspond to the situation where only one out of n oscillators is excited, the remaining ones being in their ground states, but by the symmetry of the n-oscillator state vector we do not know which of the oscillators is excited. The creation operators are defined as those which increase the number of excitations in the ensemble. This requirement alone does not completely fix the form of such operators, as shown, for example, by the freedom in the choice of the constants c_n. Different choices of c_n lead to different representations of the non-CCR algebra, but if we require that the resolution of identity property should hold not only in the one-oscillator sector, we arrive at the particular representation with $c_n = 1/\sqrt{n}$.

It is obvious that the creation operators change the number of excitations but maintain the number of oscillators. This property is unchanged if we introduce a minimally coupled interaction Hamiltonian, which leads to the important additional conservation law.

Heisenberg's original quantization of the harmonic oscillator may be intuitively regarded as a quantization of amplitude. (Having fixed frequency the only possibility of changing the energy is by changes of the amplitude of the oscillation.) Dirac's idea of quantizing the field was also based on quantization of amplitudes. For this reason one replaces classical amplitudes with operators. Averages of the operators should give classical amplitudes if evaluated at appropriate (coherent) states.

Led by the same intuitions we proceed in an analogous way. Field operators are introduced in a direct analogy to the original Dirac formulation, but now the oscillators exist in superpositions of amplitudes, polarizations *and momenta*, the latter being equivalent to superpositions of positions. This allows us to think of the field as a *single* harmonic oscillator, a single quantum object. The most general field is therefore an ensemble of such objects, in general, with an indefinite number of oscillators. The averages of field operators represent classical fields.

The free-field 4-momentum, denoted $\mathcal{P}_a$, is constructed by the standard n-particle extension of one-oscillator Hamiltonians. Similarly to other quantized systems the average energy described in terms of such a Hamiltonian is finite if both the corresponding one-particle energy of a single oscillator and the average number of oscillators are finite. However, once we have defined field operators at the indefinite-number-of-oscillators Hilbert space, we can employ standard electrodynamical definitions of 4-momentum and obtain new forms of free-field energy and momentum operators, denoted by $\boldsymbol{P}_a$. At the one-oscillator level the procedure leads us back to the 4-momentum we started with. At the multi-oscillator level, however, the resulting 4-momenta turn out to also describe processes of energy exchange between different oscillators. It is possible that such an 'interacting' 4-momentum correctly describes the dynamics of the ensemble of oscillators, but we have decided to stick to the 'non-interacting' case due to its simplicity.

Nevertheless, with an appropriate definition of creation and annihilation operators (with $c_n = 1/\sqrt{n}$) one can guarantee that the two free-field 4-momenta have averages differing only by the (finite) vacuum contribution. The same choice of creation operators implies that the right-hand side of the non-CCR algebra contains operators (POV measures) satisfying a resolution of identity.

All of the above properties of the new formalism were, so to say, put in by hand on the basis of our understanding of the physical structure we had in mind. Some unexpected properties of the formalism were revealed by concrete perturbative calculations. In particular, it was not originally anticipated that our description of the vacuum state may additionally regularize the perturbation theory by introducing a natural (in general, smooth and vacuum-dependent) cut-off. This is related to the fact that oscillators with high frequencies are less probable since otherwise the vacuum state would not be square-integrable.

The fact that such a regularization necessarily occurs leads to the possibility of experimental tests of the non-CCR formalism. In particular, the ratio of probabilities of one-photon and two-identical-photons spontaneous emissions differs by a factor from the CCR case. On the other hand, appropriate ratios of probabilities of single-photon and two-different-photons spontaneous emissions are expected to be the same as in the canonical theory. We cannot predict an exact value of the ratio since it is vacuum-dependent, but under some assumptions concerning the vacuum state we can produce at least rough estimates. An experimental verification that the proportionality factor differs from unity could give an information concerning the form of the vacuum state.

Another consequence of the construction is the conservation of the number of oscillators. This additional conservation law has implications for the statistics since one has to introduce a non-vanishing chemical potential $\mu < 0$. It is only due to this potential that we recover the Planck law for temperatures $T \ll T_0$, where T_0 is a constant. For $T \gg T_0$ one expects a distribution of a $\mu = 0$ type, which is similar to the Planck one but with lowered maximum.

9. Final remarks

'A theory that is as spectacularly successful as quantum electrodynamics has to be more or less correct, although we may not be formulating it in just the right way'. This quotation from Weinberg [12] could serve as a motto for opening our paper.

Let us close the paper with another quotation from Janes [5]:

> Present quantum electrodynamics contains many very important 'elements of truth', but also some clear 'elements of nonsense'. Because of the divergences and ambiguities, there is general agreement that a rather deep modification of the theory is needed, but in some 40 years of theoretical work, nobody has seen how to disentangle the truth from the nonsense. In such a situation, one needs more experimental evidence, but during that same 40 years we have found no clues from the laboratory as to what specific features of QED might be modified. Even worse, in the absence of any alternative theory whose predictions differ from those of QED in known ways, we have no criterion telling us *which* experiments would be relevant ones to try.
>
> It seems useful, then, to examine the various disturbing features of QED, which give rise to mathematical or conceptual difficulties, to ask whether present empirical evidence demands their presence, and to explore the consequences of the modified (although perhaps rather crude and incomplete) theories in which these features are removed. Any difference between the predictions of QED and some alternative theory, corresponds to an experiment which might distinguish between them; if it appears untried but feasible, then we have the opportunity to subject QED to a new test in which we know just what to look for, and which we would be very unlikely to think of without the alternative theory. For this purpose, the alternative theory need not be worked out as completely as QED; it is sufficient if we know in what way their predictions will differ in the area of interest. Nor does the alternative theory need to be free of defects in all other respects; for if experiment should show that it contains just a single 'element of truth' that is *not* in QED, then the alternative theory will have served its purpose; we would have the long-missing clue showing in what way QED must be modified, and electrodynamics (and, I suspect, much more of theoretical physics along with it) could get moving again.

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